Camera calibration with moving one-dimensional objects

F.C. Wu, Z.Y. Hu*, H.J. Zhu

National Laboratory of Pattern Recognition, Institute of Automation Chinese Academy of Sciences, P.O. Box 2728, Beijing 100080, PR China

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Abstract

In this paper, we show that the rotating 1D calibrating object used in the literature is in essence equivalent to a familiar 2D planar calibration object. In addition, we also show that when the 1D object undergoes a planar motion rather than rotating around a fixed point, such equivalence still holds but the traditional way fails to handle it. Experiments are carried out to verify the theoretical correctness and numerical robustness of our results.

Keywords: Camera calibration; 1D calibration object; Homography; Vanishing points; Absolute points

1. Introduction

Camera calibration is a necessary step to extract metric information from 2D images. According to the dimension of the used calibration object, the camera calibration can be classified as 3D object-based calibration [1–4], 2D planar object-based calibration [5–7], 1D segment-based calibration [8], and self-calibration, or 0D calibration [9–16], since it does not use any physical calibration objects and only correspondences of image points are involved.

Although self-calibration, 3D object-based calibration and 2D object-based calibration have been extensively studied in the literature, to our knowledge, for 1D object-based calibration there exists only Zhang’s result [8] that: If 1D calibration object has three or more collinear points with known distances, then camera calibration problem can be solved from images of the calibration object rotating around a known fixed point.

In this paper, we show that the 1D calibration object of a rotating segment around a fixed point in Zhang’s setup is in essence equivalent to a 2D rectangular calibration object with unknown sides. In other words, although the calibration object is 1D, we can find four images of an “imaginary rectangle’s 4 corners” when the 1D object rotates around a fixed point. By such an equivalence, the calibration constraints as well as the underlying concept can be substantially simplified. In addition, we show that rather than rotating around a fixed point, if the 1D-object undergoes a general planar motion, the above 1D object-based calibration method remains valid. In other words, the planar motion case can be converted to a rotating one at first, then the method for rotating one can be used to calibrate the camera.

The paper is organized as follows. Some preliminaries, such as the camera model, homography, image of the absolute conic and vanishing points of orthogonal directions are introduced in Section 2. Zhang’s method is outlined in Section 3. The equivalence of calibration objects is elaborated in Section 4. Section 5 is about extensions of object motion. Experiments are reported in Section 6.

In this paper, a 2D point is denoted by \( \mathbf{a} = [u, v]^T \), a 3D point by \( \mathbf{A} = [X, Y, Z]^T \). The corresponding homogeneous vector is denoted by \( \mathbf{a} = [u, v, 1]^T \) and \( \mathbf{A} = [X, Y, Z, 1]^T \), respectively.

* Corresponding author. Tel.: +86 10 626 16 540; fax: +86 10 625 51 993.
E-mail address: huzy@nlpr.ia.ac.cn (Z.Y. Hu).
2. Preliminaries

2.1. Camera model

A camera is modeled by the usual pinhole, then a 3D point $A$ is projected to its image point $a$ by

$$
\mathbf{a} = K[R \ t] \tilde{A}, \quad K = \begin{bmatrix} f_u & s & u_0 \\ 0 & f_v & v_0 \\ 0 & 0 & 1 \end{bmatrix},
$$

where $z$ is a scale factor (projection depth of 3D point $A$), $P = K[R \ t]$ the camera matrix, $[R \ t]$, extrinsic matrix, that is, the rotation and translation from the world frame to the camera frame, and $K$ the camera intrinsic matrix, with $f_u, f_v$ the scale factors in the image $u$ and $v$ axes, $(u_0, v_0)$ the principal point, and $s$ the parameter describing the skew of the two image axes. The task of camera calibration is to determine these five intrinsic parameters.

2.2. Homography

Homography is a non-singular $3 \times 3$ matrix, which defines a homogeneous linear transformation from a plane to another in projective space. Let the $xy$-plane of the world frame coincide with a space plane $\pi$, then the homography $H$ from plane $\pi$ to the image plane can be expressed as

$$
H = K[r_1 \ r_2 \ t],
$$

where $K$ is the intrinsic matrix, $[r_1 \ r_2 \ r_3 \ t]$ the extrinsic matrix.

Let $\pi_\infty$ be the plane at infinity in 3D space, then the homography from $\pi_\infty$ to the image plane, called the infinite homography, is given by

$$
H = K[r_1 \ r_2 \ r_3]. \quad (2.1)
$$

2.3. The image of the absolute conic and the vanishing points of orthogonal directions

The absolute conic $\Omega_\infty$ is a virtual point conic on the infinite plane $\pi_\infty$. It consists of points $\mathbf{X}=[x_1, x_2, x_3, x_4]^T$ such that

$$
x_1^2 + x_2^2 + x_3^2 = 0,
$$

or

$$
\mathbf{X}_\infty^T \mathbf{X}_\infty = 0, \quad (\mathbf{X}_\infty \in \pi_\infty).
$$

So, the absolute conic can be expressed as $\Omega_\infty = E$ (the identity matrix), and it is invariant under any 3D similarity transformation.

Let $C$ be the image of the absolute conic $\Omega_\infty$, and camera matrix $P = K[R \ t]$. Since the infinite homography from $\pi_\infty$ to the image plane is $H = KR$, we have

$$
C = H^{-T} \Omega_\infty H^{-1} = K^{-T} K^{-1}
$$

Thus, the image of $\Omega_\infty$, $K^{-T} K^{-1}$, is a conic on the image plane, which depends only on the camera’s intrinsic parameters.

The infinite point of a line defines its direction. Let $p_\infty^{(1)}, p_\infty^{(2)}$, be respectively, the infinite points of two orthogonal lines, then we have $(p_\infty^{(1)})^T \Omega_\infty p_\infty^{(2)} = 0$ since two orthogonal directions is a pair of conjugate points with respect to the absolute conic. Thus, image points $\tilde{p}_\infty^{(1)}, \tilde{p}_\infty^{(2)}$ of $p_\infty^{(1)}, p_\infty^{(2)}$, called vanishing points of orthogonal directions, satisfy the following equation:

$$
(\tilde{p}_\infty^{(1)})^T K^{-T} K^{-1} \tilde{p}_\infty^{(2)} = 0. \quad (2.2)
$$

3. Zhang’s work

Since our work is closely related to Zhang’s work in [8], an outline of Zhang’s work is necessary at first.

3.1. 1D calibration object

Let the length of a line segment be $\|A - B\| = L$. If the line segment $AB$ contains some points, say $C, D, E, F$, and the distance between any two of them is known, then the line segment $AB$ is called a 1D calibration object.

It is shown in [8] that a 1D calibration object should contain at least three known points, and more points do not provide any new independent constraint on the intrinsic parameters, but can boost the calibration’s robustness in practice because data redundancy can combat the noise in image data. In this paper, we only consider the minimal configuration composed of three collinear points, but it is straightforward to extend the result if 1D calibration object has four or more collinear points.

Assume 1D calibration object has three points, say $A, B, C$. Since the distances from $C$ to $A, B$ are known, point $C$ can be expressed as

$$
C = \lambda_A A + \lambda_B B, \quad (3.1)
$$

where $\lambda_A, \lambda_B$ are known scalars, e.g. if $C$ is the midpoint of line segment $AB$, then $\lambda_A = \lambda_B = \frac{1}{2}$.

For the convenience of statement, 1D calibration object is also said to be the line segment $AB$ hereinafter, and the line defined by two points $C, D$ is called the line $CD$, the plane defined by two coplanar line segments $CD, EF$ is denoted by $\pi(\mathbf{CD}, EF)$.

3.2. Zhang’s method

Zhang in [8] showed that the intrinsic parameters can be determined from images of a 1D calibration object rotating around a known fixed point if the calibration object has three or more known points.

As shown in Fig. 1, point $A$ is the fixed point in space, and the line segment $AB$ rotates $N$ times around $A$. The
Eq. (3.5) is equivalent to
\[ z_A^2 (h(i))^T K^{-T} (h(i)) = L^2. \]

Hence, the following constraint equation can be obtained on the intrinsic parameters:
\[ (h(j))^T K^{-T} K^{-1} h(j) - (h(0))^T K^{-T} K^{-1} h(0) = 0, \]
\[ 1 \leq j \leq N. \]

Since five intrinsic parameters should be calibrated, the problem can be solved from Eq. (3.6) when \( N \geq 5 \). Eq. (3.6) is the constraint used in Zhang’s work [8].

Note that when \( 1 \leq i < j \leq N \), although the following constraint can be similarly derived,
\[ (h(j))^T K^{-T} K^{-1} h(j) - (h(i))^T K^{-T} K^{-1} h(i) = 0. \]

In fact, it does not constitute a new independent constraint with respect to those in Eq. (3.6) since it is simply a linear combination of the following equations:
\[ (h(j))^T K^{-T} K^{-1} h(j) - (h(0))^T K^{-T} K^{-1} h(0) = 0, \]
\[ (h(j))^T K^{-T} K^{-1} h(j) - (h(0))^T K^{-T} K^{-1} h(0) = 0. \]

4. Our geometrical method

In this section, the equivalence of a rotating 1D calibration object to a 2D rectangle with unknown sides will be discussed.

4.1. Equivalence of calibrating objects

The following equivalence proposition is the theoretical foundation of our geometrical method.

**Proposition 4.1.** Let line-segment \( AB \) rotate around point \( A \), assume that the positions of \( B, C \) before and after rotation are \( (B(0), C(0)) \) and \( (B(j), C(j)) \), and their image points \( (b(0), c(0)), (b(j), c(j)) \), respectively, let image point of \( A \) be \( \tilde{a} \), then we can obtain the images of two “imaginary rectangles \( B(0)iB(j)B(0)(j)B(j)\) and \( C(0)iC(j)C(0)(j)C(j)\)” in the image plane. In other words, the line-segment AB rotating at one time is equivalent to the two 2D rectangular objects \( B(0)iB(j)B(0)(j)B(j)\) and \( C(0)iC(j)C(0)(j)C(j)\).

**Proof.** As shown in Fig. 2, points \( B(0), C(0), B(j), C(j) \), are, respectively, the symmetrical point of \( B(0), C(0), B(j), C(j) \) about point \( A \). Evidently, \( [C(0), C(j), C(0), C(j)] \) and \( [B(0), B(j), B(0), B(j)] \) are the corners of two rectangles \( C(0)iC(j)C(0)(j)C(j)\) and \( B(0)iB(j)B(0)(j)B(j)\), respectively, the two rectangles have the same center \( A \), and
\[ C(0)C(j)i/C(0)iC(j)/B(0)iB(j)/B(0)(j)B(j). \]
In order to prove this proposition, we need only to determine the image points \( \tilde{e}^{(i)}, \tilde{e}^{(j)}, \tilde{b}^{(i)}, \tilde{b}^{(j)} \) of points \( C^{(i)}, C^{(j)}, B^{(i)}, B^{(j)} \).

1. Determining the image point \( \tilde{p}^{(0)}(\tilde{p}^{(j)}) \) of the infinite point \( P_{\infty}^{(0)}(P_{\infty}^{(j)}) \) on space line \( AB^{(0)}(AB^{(j)}) \), i.e., to determine the vanishing point of space line \( AB^{(0)}(AB^{(j)}) \).

Let \( \lVert A - C^{(0)} \rVert = \lVert A - C^{(j)} \rVert = d_1, \lVert B^{(0)} - C^{(0)} \rVert = \lVert B^{(j)} - C^{(j)} \rVert = d_2 \), both are known. Since the cross-ratio of four points \( \{A, B^{(0)}, C^{(0)}, P_{\infty}^{(0)}\} \) is equal to the simple ratio of three points \( \{A, B^{(0)}, C^{(0)}\} \), we have

\[
\text{Cross}(A, B^{(0)}; C^{(0)}, P_{\infty}^{(0)}) = -d_2/d_1.
\]

The cross-ratio is invariant under any projective transformation, therefore we have

\[
\text{Cross}(\tilde{a}, \tilde{b}^{(0)}; \tilde{c}^{(0)}, \tilde{p}^{(0)}) = -d_2/d_1. \tag{4.1}
\]

Hence, we can determine the image point \( \tilde{p}^{(0)} \) of \( P_{\infty}^{(0)} \). Similarly the image point \( \tilde{p}^{(j)} \) of \( P_{\infty}^{(j)} \) can be determined.

2. Determining \( \tilde{c}^{(0)}, \tilde{c}^{(j)}, \tilde{b}^{(0)}, \tilde{b}^{(j)} \).

Since \( C^{(0)} \) is the symmetrical point of \( C^{(0)} \) about \( A \), then

\[
\text{Cross}(C^{(0)}, C^{(0)}; A, P_{\infty}^{(0)}) = -1.
\]

Therefore, we have

\[
\text{Cross}(\tilde{c}^{(0)}, \tilde{c}^{(0)}; \tilde{a}, \tilde{p}^{(0)}) = -1.
\]

Hence, \( \tilde{c}^{(0)} \) can be determined because \( \tilde{p}^{(0)} \) is known here. Similarly we can also determine \( \tilde{c}^{(j)}, \tilde{b}^{(0)}, \tilde{b}^{(j)} \). □

**Computing** \( \tilde{p}^{(0)}, \tilde{p}^{(j)}, \tilde{c}^{(i)}, \tilde{c}^{(j)}, \tilde{b}^{(i)}, \tilde{b}^{(j)} \): Given four collinear points \( \tilde{x}_j = (x_j, y_j, 1)^T (j = 1, 2, 3, 4) \) the cross-ratio is defined as

\[
\text{Cross}(\tilde{x}_1, \tilde{x}_2; \tilde{x}_3, \tilde{x}_4) = \begin{vmatrix} x_3 - x_1 & x_4 - x_1 \\ x_3 - x_2 & x_4 - x_2 \\ y_3 - y_1 & y_4 - y_1 \\ y_3 - y_2 & y_4 - y_2 \end{vmatrix},
\]

Then, from Eq. (4.1), we have

\[
\begin{align*}
x_2(0) - x_\tilde{a} & = \frac{x_p(0) - x_\tilde{a}}{x_2(0) - x_{\tilde{b}(0)}} \cdot \frac{y_2(0) - y_\tilde{a}}{y_2(0) - y_{\tilde{b}(0)}} \cdot \frac{y_p(0) - y_\tilde{a}}{y_p(0) - y_{\tilde{b}(0)}} \cdot \frac{y_p(0) - y_\tilde{a}}{y_p(0) - y_{\tilde{b}(0)}} \cdot \frac{y_p(0) - y_\tilde{a}}{y_p(0) - y_{\tilde{b}(0)}} = -d_2/d_1.
\end{align*}
\]

By solving the above equations, we obtain

\[
\begin{align*}
\tilde{p}^{(0)} &= [x_p(0), y_p(0), 1]^T \\
&= \left[ \frac{d_2x_\tilde{a}(x_2(0) - x_{\tilde{b}(0)}) + d_1x_{\tilde{a}}(x_2(0) - x_\tilde{a})}{d_2(x_2(0) - x_{\tilde{b}(0)}) + d_1(x_2(0) - x_\tilde{a})}, \\
&\quad \frac{d_2y_\tilde{a}(y_2(0) - y_{\tilde{b}(0)}) + d_1y_{\tilde{a}}(y_2(0) - y_\tilde{a})}{d_2(y_2(0) - y_{\tilde{b}(0)}) + d_1(y_2(0) - y_\tilde{a})} \right]^T. \tag{4.2a}
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
\tilde{p}^{(j)} &= [x_p(j), y_p(j), 1]^T \\
&= \left[ \frac{x_\tilde{a}(x_2(j) - x_{\tilde{b}(j)}) + d_1x_{\tilde{a}}(x_2(j) - x_\tilde{a})}{x_2(j) - x_{\tilde{b}(j)} + d_1(x_2(j) - x_\tilde{a})}, \\
&\quad \frac{y_\tilde{a}(y_2(j) - y_{\tilde{b}(j)}) + d_1y_{\tilde{a}}(y_2(j) - y_\tilde{a})}{y_2(j) - y_{\tilde{b}(j)} + d_1(y_2(j) - y_\tilde{a})} \right]^T. \tag{4.2b}
\end{align*}
\]

**Cross** \( \tilde{c}^{(i)}, \tilde{c}^{(j)}; \tilde{b}^{(i)}, \tilde{b}^{(j)} \) : Given four collinear points \( \tilde{y}_j = (x_j, y_j, 1)^T (j = 1, 2, 3, 4) \) the cross-ratio is defined as

\[
\text{Cross}(\tilde{x}_1, \tilde{x}_2; \tilde{x}_3, \tilde{x}_4) = \begin{vmatrix} x_3 - x_1 & x_4 - x_1 \\ x_3 - x_2 & x_4 - x_2 \\ y_3 - y_1 & y_4 - y_1 \\ y_3 - y_2 & y_4 - y_2 \end{vmatrix},
\]

\[
\begin{align*}
\text{Cross}(\tilde{x}_1, \tilde{x}_2; \tilde{x}_3, \tilde{x}_4) &= \text{Cross}(\tilde{y}_1, \tilde{y}_2; \tilde{y}_3, \tilde{y}_4) \\
&= \text{Cross}(\tilde{y}_1, \tilde{y}_2; \tilde{y}_3, \tilde{y}_4),
\end{align*}
\]

\[
\begin{align*}
\tilde{b}^{(i)} &= [x_{\tilde{b}(i)}, y_{\tilde{b}(i)}, 1]^T \\
&= \left[ \frac{x_\tilde{a}(x_{\tilde{b}(i)} - x_{\tilde{b}(i)}) + d_1x_{\tilde{a}}(x_{\tilde{b}(i)} - x_\tilde{a})}{x_{\tilde{b}(i)} - x_{\tilde{b}(i)} + d_1(x_{\tilde{b}(i)} - x_\tilde{a})}, \\
&\quad \frac{y_\tilde{a}(y_{\tilde{b}(i)} - y_{\tilde{b}(i)}) + d_1y_{\tilde{a}}(y_{\tilde{b}(i)} - y_\tilde{a})}{y_{\tilde{b}(i)} - y_{\tilde{b}(i)} + d_1(y_{\tilde{b}(i)} - y_\tilde{a})} \right]^T. \tag{4.2c}
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
\tilde{b}^{(j)} &= [x_{\tilde{b}(j)}, y_{\tilde{b}(j)}, 1]^T \\
&= \left[ \frac{x_\tilde{a}(x_{\tilde{b}(j)} - x_{\tilde{b}(j)}) + d_1x_{\tilde{a}}(x_{\tilde{b}(j)} - x_\tilde{a})}{x_{\tilde{b}(j)} - x_{\tilde{b}(j)} + d_1(x_{\tilde{b}(j)} - x_\tilde{a})}, \\
&\quad \frac{y_\tilde{a}(y_{\tilde{b}(j)} - y_{\tilde{b}(j)}) + d_1y_{\tilde{a}}(y_{\tilde{b}(j)} - y_\tilde{a})}{y_{\tilde{b}(j)} - y_{\tilde{b}(j)} + d_1(y_{\tilde{b}(j)} - y_\tilde{a})} \right]^T. \tag{4.2d}
\end{align*}
\]
4.2. The resulting constraints from the imaginary 2D calibrating object

The space lines \( \{C(0)C(j), C'(0)C(j), B(0)B'(j), B'(0)B(j)\} \) are parallel to each other, therefore their image lines \( \{c(0)c'(j), c'(0)c(j), b(0)b'(j), b'(0)b(j)\} \) intersect the line \( p(0)p(j) \) at the same point, denoted by \( \tilde{v}_{ij}^{(0)} \). Similarly, the image lines \( \{c(0)c'(j), c'(0)c(j), b(0)b'(j), b'(0)b(j)\} \) intersect also the line \( p(0)p(j) \) at the same point, denoted by \( \tilde{v}_{ij}^{(0)} \).

Since \( \tilde{v}_{ij}^{(0)} \) and \( \tilde{v}_{ij}^{(0)} \) are vanishing points of a pair of orthogonal directions, according to Eq. (2.2), we obtain the following constraint equations on the intrinsic parameters:

\[
\left( \tilde{v}_{ij}^{(0)} \right)^T K^{-T} K^{-1} \tilde{v}_{ij}^{(0)} = 0, \quad 1 \leq i < j \leq N. \tag{4.3}
\]

Thus, if the line segment \( AB \) rotates \( N \) times around \( A \), we can obtain \( N \) constraint equations on the intrinsic parameters, and the camera calibration problem can be solved from linear equations (4.3) when \( N \geq 5 \).

Note that when \( 1 \leq i < j \leq N \), although the following constraint can be similarly derived:

\[
\left( \tilde{v}_{ij}^{(0)} \right)^T K^{-T} K^{-1} \tilde{v}_{ij}^{(0)} = 0, \quad 1 \leq i < j \leq N.
\]

They do not constitute new independent constraints with respect to those in Eq. (4.3) since they can be produced by the following equations:

\[
\begin{align*}
\left( \tilde{v}_{ij}^{(0)} \right)^T K^{-T} K^{-1} \tilde{v}_{ij}^{(0)} & = 0, \\
\left( \tilde{v}_{ij}^{(0)} \right)^T K^{-T} K^{-1} \tilde{v}_{ij}^{(0)} & = 0.
\end{align*}
\]

**Computing \( \tilde{v}_{ij}^{(0)} \), \( \tilde{v}_{ij}^{(0)} \):** In order to compute \( \tilde{v}_{ij}^{(0)} \), let

\[
\begin{align*}
I_1^{(0)} & = c^{(0)} \times c^{(j)} = \left[ a_1^{(0)} b_1^{(0)} c_1^{(0)} \right]^T, \\
I_2^{(0)} & = c^{(0)} \times c^{(j)} = \left[ a_2^{(0)} b_2^{(0)} c_2^{(0)} \right]^T, \\
I_3^{(0)} & = b^{(0)} \times b^{(j)} = \left[ a_3^{(0)} b_3^{(0)} c_3^{(0)} \right]^T, \\
I_4^{(0)} & = b^{(0)} \times b^{(j)} = \left[ a_4^{(0)} b_4^{(0)} c_4^{(0)} \right]^T, \\
I_5^{(0)} & = p^{(0)} \times p^{(j)} = \left[ a_5^{(0)} b_5^{(0)} c_5^{(0)} \right]^T.
\end{align*}
\]

They are five image lines. Theoretically, these lines intersect at the same point \( \tilde{v}_{ij}^{(0)} \), i.e., \( \tilde{v}_{ij}^{(0)} = I_1^{(0)} \times I_2^{(0)}, \quad 1 \leq k < l \leq 5 \). However, no three lines exactly intersect at the same point in practice due to noise in image data, some robust method, for example, the least-squares method can be used to compute \( \tilde{v}_{ij}^{(0)} \). Similarly, we can compute \( \tilde{v}_{ij}^{(0)} \).

Our method is called a geometrical one because our calibration constraints (4.3) are from proper geometrical concepts, such as parallelism, orthogonality, etc.

**4.3. The relationship between our geometric method and Zhang’s method**

In this section, we show that the constraints in Eq. (4.3) produced by our geometric method are equivalent to Zhang’s constraints in Eq. (3.5).

**Proposition 4.2.** When line-segment \( AB \) rotates \( N \) times around \( A \), the constraint equations produced by Zhang’s method

\[
\left( h^{(0)} \right)^T K^{-T} K^{-1} h^{(0)} - \left( h^{(0)} \right)^T K^{-T} K^{-1} h^{(0)} = 0,
\]

\[
1 \leq i \leq N
\]

are equivalent to the constraint equations produced by our geometric method:

\[
\left( \tilde{v}_{ij}^{(0)} \right)^T K^{-T} K^{-1} \tilde{v}_{ij}^{(0)} = 0, \quad 1 \leq j \leq N.
\]

**Proof.** As shown in Fig. 3,

\[
K^{-1} \tilde{v}_{ij}^{(0)} = P_{1n}^{(0)} = \eta_1^{(0)} P_{1n}^{(0)},
\]

\[
K^{-1} \tilde{v}_{ij}^{(0)} = P_{2n}^{(0)} = \eta_2^{(0)} P_{2n}^{(0)}
\]

where

\[
\eta_k^{(0)} = \frac{\|P_{nk}^{(0)}\|}{\|P_{nk}^{(0)}\|}, \quad k = 1, 2.
\]

Then we have

\[
\left( \tilde{v}_{ij}^{(0)} \right)^T K^{-T} K^{-1} \tilde{v}_{ij}^{(0)} = \eta_1^{(0)} \eta_2^{(0)} \left( P_{1n}^{(0)} P_{2n}^{(0)} \right)^T P_{1n}^{(0)} \cdot \eta_2^{(0)}.
\]

According to Proposition 4.1, \( P_{1n}^{(0)}, P_{2n}^{(0)} \) are two orthogonal directions of the rectangle \( \{B(0), B'(j), B''(0), B(j)\} \).
Let $\theta_{0j}$ be the unknown rotation angle between lines $AB^{(0)}$ and $AB^{(j)}$, then we have

\[
B^{(0)} - A = L((\cos(\theta_{0j}/2) \cdot P_{n1}^{(j)}) + \sin(\theta_{0j}/2) \cdot P_{n2}^{(j)}),
\]

\[
B^{(j)} - A = L((\cos(\theta_{0j}/2) \cdot P_{n1}^{(0)}) - \sin(\theta_{0j}/2) \cdot P_{n2}^{(0)}),
\]

also,

\[
(A - B^{(0)})^T(A - B^{(0)}) = L^2(\cos^2(\theta_{0j}/2) \cdot (P_{n1}^{(j)})^T P_{n1}^{(j)})
+ 2 \cos(\theta_{0j}/2) \sin(\theta_{0j}/2) \cdot (P_{n1}^{(j)})^T P_{n2}^{(j)}
+ \sin^2(\theta_{0j}/2) \cdot (P_{n2}^{(j)})^T P_{n2}^{(j)}),
\]

\[
(A - B^{(j)})^T(A - B^{(j)}) = L^2(\cos^2(\theta_{0j}/2) \cdot (P_{n1}^{(0)})^T P_{n1}^{(0)})
- 2 \cos(\theta_{0j}/2) \sin(\theta_{0j}/2) \cdot (P_{n1}^{(0)})^T P_{n2}^{(0)}
+ \sin^2(\theta_{0j}/2) \cdot (P_{n2}^{(0)})^T P_{n2}^{(0)}).
\]

So, we have

\[
(A - B^{(j)})^T(A - B^{(j)}) - (A - B^{(0)})^T(A - B^{(0)}) = 4L^2 \cos(\theta_{0j}/2) \sin(\theta_{0j}/2) \cdot (P_{n1}^{(j)})^T P_{n2}^{(j)}. \tag{4.5}
\]

According to the discussion in Section 3.2, we obtain

\[
(A - B^{(0)})^T(A - B^{(0)}) = z_A^2 h_0^T K^{-T} K^{-1} h_0^T),
\]

\[
(A - B^{(j)})^T(A - B^{(j)}) = z_A^2 h_0^+(j)^T K^{-T} K^{-1} h_0^+(j).
\]

By Eq. (4.5), we have

\[
(h^{(j)})^T K^{-T} K^{-1} (h^{(j)}) - (h^{(0)})^T K^{-T} K^{-1} (h^{(0)}) = 4L^2 \cos(\theta_{0j}/2) \sin(\theta_{0j}/2) \cdot (P_{n1}^{(j)})^T P_{n2}^{(j)}. \tag{4.6}
\]

Then, by Eqs. (4.4) and (4.6), we obtain

\[
(h^{(j)})^T K^{-T} K^{-1} (h^{(j)}) - (h^{(0)})^T K^{-T} K^{-1} (h^{(0)}) = 4L^2 \cos(\theta_{0j}/2) \sin(\theta_{0j}/2) \cdot (V_1^{(0)})^T K^{-T} K^{-1} V_2^{(0)}. \tag{4.7}
\]

Therefore, equations

\[
(h^{(i)})^T K^{-T} K^{-1} (h^{(i)}) - (h^{(0)})^T K^{-T} K^{-1} (h^{(0)}) = 0, \quad 1 \leq i \leq N,
\]

are equivalent to equations

\[
(V_1^{(0)})^T K^{-T} K^{-1} V_2^{(0)} = 0, \quad 1 \leq j \leq N. \tag{5.1}
\]

5. An extension of 1D-object motion

In the preceding section, the 1D calibration object is assumed to rotate about a fixed point. In this section, we show that if the object is under a 2D planar motion, the calibration principle still holds. In particular, we have the following proposition.

**Proposition 5.1.** Assume the motion of line-segment $AB$ is a general rigid planar motion in space, let $(A^{(0)}, B^{(0)}, C^{(0)})$ and $(A^{(j)}, B^{(j)}, C^{(j)})$ be the positions of points $A, B, C$ before and after the motion, respectively, and their image points be $\tilde{a}^{(0)}, \tilde{b}^{(0)}, \tilde{c}^{(0)}$ and $\tilde{a}^{(j)}, \tilde{b}^{(j)}, \tilde{c}^{(j)}$, respectively, then if the line-segment $A^{(0)}B^{(0)}$ is not parallel to $A^{(j)}B^{(j)}$, i.e., if the line-segment $AB$ is not under a pure translation, then we can obtain the images of two imaginary rectangles.

**Proof.** As shown in Fig. 4, in plane $\pi(A^{(0)}B^{(0)}, A^{(j)}B^{(j)})$, we translate the line segment $A^{(0)}B^{(0)}$ such that $A^{(0)}$ coincides with $A^{(j)}$. Let $B^{(0)}_t$, $C^{(0)}_t$ be the position of $B^{(0)}$, $C^{(0)}$ after the translation. The space line $A^{(j)}B^{(j)}$ does not coincide with $B^{(0)}_tC^{(0)}_t$ since the space line $A^{(0)}B^{(0)}$ is not parallel to line $A^{(j)}B^{(j)}$. We need only to determine the image points $\tilde{b}^{(0)}_t$, $\tilde{c}^{(0)}_t$ of $B^{(0)}_t$, $C^{(0)}_t$ in order to prove this proposition. Next, we show how to determine the image point $\tilde{b}^{(0)}_t$.

From the proof of Proposition 4.1, we can determine the vanishing point $\tilde{p}^{(0)}(\tilde{p}^{(j)})$ of the space line $A^{(0)}B^{(0)}(A^{(j)}B^{(j)})$. Thus, the vanishing line $\tilde{p}^{(0)}(\tilde{p}^{(j)})$ of the plane $\pi(A^{(0)}B^{(0)}, A^{(j)}B^{(j)})$ can be computed. The image point $\tilde{p}^{(0)}$ is also the vanishing point of space line $A^{(j)}B^{(j)}_t$ since $A^{(j)}B^{(j)}_t//A^{(0)}B^{(0)}$. So, the line $\tilde{a}^{(j)}(\tilde{p}^{(0)})$ is the image line of space line $A^{(j)}B^{(j)}_t$. Evidently, the intersecting point of two lines $[\tilde{a}^{(0)}_t, \tilde{b}^{(0)}_t, \tilde{c}^{(0)}_t]$, denoted by $\tilde{p}$, is the vanishing point of space line $A^{(0)}_tA^{(j)}$. The point $\tilde{p}$ is also the vanishing point of line $B^{(0)}_tB^{(j)}_t$ and line $C^{(0)}_tC^{(j)}_t$ since $A^{(0)}_tA^{(j)}/B^{(0)}_tB^{(j)}_t/C^{(0)}_tC^{(j)}_t$. Thus, the image line of space line $B^{(0)}_tB^{(j)}_t(C^{(0)}_tC^{(j)}_t)$ is the line $\tilde{b}^{(0)}_t\tilde{p}^{(0)}(\tilde{p}^{(j)}\tilde{p}^{(j)})$. Hence, the image point $\tilde{b}^{(0)}_t$ of $B^{(j)}_t$ is the intersecting point of two lines $[\tilde{b}^{(0)}_t, \tilde{a}^{(j)}(\tilde{p}^{(0)})]$ since $\tilde{b}^{(0)}_t$ is the intersecting point of two space lines $[B^{(0)}_tB^{(j)}_t(A^{(j)}B^{(j)}_t)]$. Similarly, the image point $\tilde{c}^{(0)}_t$ of $C^{(j)}_t$ is the intersecting point of two lines $[\tilde{c}^{(0)}_t, \tilde{a}^{(j)}(\tilde{p}^{(0)})]$. 

**Computing $\tilde{p}$, $\tilde{b}^{(0)}_t$, $\tilde{c}^{(0)}_t$.** By Eqs. (4.2a) and (4.2b), we can compute $\tilde{p}^{(0)}, \tilde{p}^{(j)}$. Since $\tilde{p}$ is the intersecting point of two lines $[\tilde{a}^{(0)}_t, \tilde{a}^{(j)}], [\tilde{p}^{(0)}], [\tilde{p}^{(j)}]$, we have

\[
\tilde{p} = (\tilde{a}^{(0)} \times \tilde{a}^{(j)}) \times (\tilde{p}^{(0)} \times \tilde{p}^{(j)}). \tag{5.1}
\]

Similarly, we can obtain

\[
\tilde{b}^{(0)}_t = (\tilde{b}^{(0)} \times \tilde{p}) \times (\tilde{a}^{(j)} \times \tilde{p}^{(j)}). \tag{5.2}
\]

\[
\tilde{c}^{(0)}_t = (\tilde{c}^{(0)} \times \tilde{p}) \times (\tilde{a}^{(j)} \times \tilde{p}^{(j)}). \tag{5.3}
\]
5.1. The constraint equations

Since we can obtain the image points $\tilde{a}(j)$, $\tilde{b}(j)$, $\tilde{c}(j)$ of the line-segment $AB$ rotating equivalently around point $A$, by the Proposition 4.1, the vanishing points $\tilde{v}_1(j)$, $\tilde{v}_2(j)$ of two orthogonal directions can be computed, hence we obtain the constraint equations on the intrinsic parameters as

$$
(\tilde{v}_1(j))^T K^{-T} K^{-1} \tilde{v}_2(j) = 0, \quad 1 \leq j \leq N.
$$

(5.4)

Remark 5.1. Proposition 5.1 is not true if the motion of line-segment $AB$ is a pure translation. In other words, a pure translation of line-segment $AB$ does not give rise to any constraints on the intrinsic parameters.

Remark 5.2. If the positions of line-segment $AB$ before and after a general rigid motion are not coplanar, no constraints on the intrinsic parameters is possible unless the two positions are orthogonal each other.

5.2. Motion recovering

Let $\theta_{ij}$ be the rotation angle between lines $A^{(0)}B^{(0)}$ and $A^{(j)}B^{(j)}$, then we have

$$
\cos \theta_{ij} = \frac{(\tilde{p}(j))^T K^{-T} K^{-1} \tilde{p}(j)}{\sqrt{(\tilde{p}(j))^T K^{-T} K^{-1} \tilde{p}(j)} \sqrt{(\tilde{p}(j))^T K^{-T} K^{-1} \tilde{p}(j)}},
$$

$$
1 \leq j \leq N.
$$

(5.5)

Next, we recover the motion plane $\pi(A^{(0)}B^{(0)}, A^{(j)}B^{(j)})$ and the translation vector $T^{(j)}$ of line-segment $AB$ in the camera coordinate system. We only need to compute the coordinates of points $\{A^{(0)}, B^{(0)}, A^{(j)}, B^{(j)}\}$ because once they are computed, the equation of $\pi(A^{(0)}B^{(0)}, A^{(j)}B^{(j)})$ and the translation vector $T^{(j)}$ can be computed as

$$
(\{A^{(0)} - B^{(0)}\} \times (A^{(j)} - B^{(j)}))^T (X - A^{(j)}) = 0,
$$

By Eqs. (3.3), and (3.4), we have

$$
A^{(0)} = z_{A^{(0)}} K^{-1} \tilde{a}^{(0)}, \quad B^{(0)} = z_{B^{(0)}} K^{-1} \tilde{b}^{(0)},
A^{(j)} = z_{A^{(j)}} K^{-1} \tilde{a}^{(j)}, \quad B^{(j)} = z_{B^{(j)}} K^{-1} \tilde{b}^{(j)},
$$

(5.6)

Then, by Eq. (3.5), we have

$$
z_{A^{(0)}} = L \frac{L}{\|K^{-1} h^{(0)}\|}, \quad z_{A^{(j)}} = L \frac{L}{\|K^{-1} h^{(j)}\|}
$$

(5.8)

with

$$
h^{(j)} = \tilde{a}^{(j)} + \frac{\tilde{a}^{(j)} \cdot \tilde{b}^{(j)} \cdot \tilde{c}^{(j)}}{\tilde{b}^{(j)} \cdot \tilde{c}^{(j)}} h^{(k)}.
$$

(5.9)

From Eqs. (5.6) to (5.9), the coordinates of points $\{A^{(0)}, B^{(0)}, A^{(j)}, B^{(j)}\}$ can be computed.

6. Experiments

In the case of camera rotating around a fixed point, although we have showed that our geometric method and Zhang’s method are equivalent to each other, since vanishing point determination, an intermediary step considered notoriously sensitive to noise, is involved in the geometrical
method, the numerical behavior could be different between the two methods. Hence some experimental comparisons are carried out. In the case of planar motion, since Zhang’s method cannot be applied, we only give some experimental results on the geometrical method.

6.1. Simulated data

The simulated camera’s intrinsic matrix is

\[
K = \begin{bmatrix}
1200 & 1.0 & 320 \\
0 & 1000 & 240 \\
0 & 0 & 1
\end{bmatrix}
\]

The image resolution is 700 × 700, and the length of the simulated line-segment AB is 100 cm.

6.1.1. The case of rotating around a fixed point

In the experiment, point C is the midpoint of line-segment AB and point A is fixed at \([10, 25, 120]^T\). Fifty orientations of the line-segment AB, determined by the rotation angle \((\theta, \phi)\), are sampled equidistantly for \(\theta\) in \([\pi/6, 5\pi/6]\) and \(\phi\) in \([\pi, 2\pi]\). Then points A, B, and C are projected onto the image plane.

We add Gaussian noise with 0 mean and \(\sigma\) standard deviation to the projected image points. The estimated camera parameters are compared with the ground truth, and RMS errors are measured. We vary the noise level \(\sigma\) from 0.1 to 1.0 pixel. For each noise level, 50 independent trials are performed, and the comparing results of the two methods are shown in Fig. 5. We can see that RMS errors from the two methods increase almost linearly with the noise level. The two methods are robust and efficient, but Zhang’s method is slightly better than our geometric one.

6.1.2. The case of planar motion

As in the case of a camera rotating around a fixed point, point C is the midpoint of line-segment AB. In the
Fig. 6. RMS errors from the geometric method (under planar motion): (a) parameters $f_u, f_v$, and (b) parameters $u_0, v_0$ and $s$.

experiment we choose five planes in space, and on each plane 20 orientations of the line-segment $\mathbf{AB}$ are generated at random for rotation angle $\theta$ in $[\pi/6, 5\pi/6]$ and translation vector $\mathbf{t}$ in $[10, 50] \times [10, 50]$ according to uniform distribution. Then points $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are projected onto the image plane.

Gaussian noise with 0 mean and $\sigma$ standard deviation is added to the projected image points. The estimated camera parameters are compared with the ground truth and RMS errors are measured. We vary the noise level $\sigma$ from 0.1 to 1.0 pixel. For each noise level, 50 independent trials are performed, and the results shown in Fig. 6 are RMS errors. We can see RMS errors increase almost linearly with the noise level.

6.2. Real images

For the experiment with real images, we used three beads made with plasticine and strung them together with a stick. The beads are approximately 6 cm apart, i.e. length of the stick is 12 cm. Then we move the stick on a plane. In space we choose five planes, and move the stick 15 times on each plane, a video of 75 frames are obtained. Four sample images are shown in Fig. 7. Then we detect a blob formed by image of a bead, and the centroid of each detected blob is used for camera calibration. The camera’s intrinsic matrix estimated by the geometric method is

$$
K = \begin{bmatrix}
1700.4031 & -38.9002 & 530.4339 \\
0 & 1669.3588 & 377.6818 \\
0 & 0 & 1
\end{bmatrix}.
$$
For validating the result, we reconstructed a 3D calibration object from its two images (see Fig. 8) taken by the same camera. The results shown in Fig. 9 are the reconstruction of the 3D calibration object. Fig. 9(a) displays the corners of squares on the 3D calibration object and Fig. 9(b) displays the result after texture mapping. The angle of two reconstructed planes is 89.8008° close to the ground truth 90°. The experiment shows that the geometric method is also effective for real images under planar motion.

7. Conclusions

In this paper, we proved that the rotating 1D calibrating object in the literature is in essence equivalent to a 2D rectangle with two unknown sides, and proposed a geometrical method for camera calibration accordingly. This equivalence can bring some new useful insights into the nature of 1D calibration. In addition, we also showed that when the 1D object undergoes a planar motion rather than the traditional rotation around a fixed point, such equivalence still holds but the traditional way fails to handle it.

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References


About the Author—FUCHAO WU is a professor at the Institute of Automation, Chinese Academy of Sciences. His research interests are now in computer vision, which include 3D reconstruction, active vision, and image-based modeling and rendering. E-mail address: fcwu@nlpr.ia.ac.cn.
About the Author—ZHANYI HU is a professor at the Institute of Automation, Chinese Academy of Sciences. He received his Ph.D. degree (Docteur d'Etat) in computer vision from University of Liege, Belgium, in 1993. His research interests include camera calibration and 3D reconstruction, geometric primitive extraction, vision guided robot navigation. E-mail address: huzy@nlpr.ia.ac.cn.

About the Author—HAIJIANG ZHU is currently a Ph.D. candidate at Institute of Automation, Chinese Academy of Sciences. He received his M.S. degree in 2000 from the Xi’an University of Science and Technology. His research interests include Computer Vision, 3D reconstruction, etc. E-mail address: hjzhu@nlpr.ia.ac.cn.